Hindered Settling Function with No Empirical Parameters for Polydisperse Suspensions

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Suspensions found in nature and industry generally have a distribution of particle sizes and perhaps densities, and there has been considerable interest in the sedimentation behavior of such polydisperse suspensions. The common practice in predicting sedimentation velocities of different particle species in a polydisperse suspension is to define hindered settling functions f_i and f_i^s, based either on the velocity v_i of species i relative to the bulk velocity (which is zero for batch sedimentation in a vertical container) or on the so-called slip velocity v_i^s of species i relative to the fluid:

$$v_i = u_i^o f_i, \quad v_i^s = u_i^o f_i^s \tag{1a,b}$$

where u_i^o is the settling velocity of an isolated particle of species i. Mass continuity may be used to relate v_i and v_i^s (Smith, 1966):

$$v_i = v_i^s - \sum_{j=1}^{N} v_j^s \phi_j \tag{2}$$

where ϕ_j is the volume fraction of species j, and N is the number of species representing the polydisperse suspension.

The hindered settling functions describe the reduction in particle settling velocities with increasing concentration due to interactions with surrounding particles and with the fluid upflow created by the downward settling of particles. As reviewed by Barnea and Mizrahi (1973), Garside and Al-Dibouni (1977) and Davis and Acrivos (1985), among others, considerable theoretical and experimental research has been directed at determining hindered settling functions for monodisperse suspensions. The most popular result is the simple formula of Richardson and Zaki (1954):

$$f_i = (1-\phi)^n, \quad f_i^s = (1-\phi)^{n-1} \tag{3a,b}$$

with an empirical value of $n = 5.1$ suggested by Garside and Al-Dibouni (1977) for spherical particles with low Reynolds numbers. Monodisperse hindered settling functions have been applied to polydisperse systems, using $\phi = \Sigma\phi_j$ as the total volume fraction of particles, but these do not give good agreement with experimental data for particles with large differences in size or density.

Several authors have proposed modifications to monodisperse correlations in order to better describe data from bidisperse or polydisperse systems (for example, Mirza and Richardson, 1979; Masliyah, 1979; Selim et al., 1983; Patwardhan and Tien, 1985). For example, the formula of Selim et al. (1983) includes the Richardson-Zaki correlation for the slip velocity, multiplied by a correction function which replaces the fluid density with the mixture density of fluid plus slower-settling particles:

$$f_i^s = (1-\phi)^{n-1}(\rho_i - \rho_m)/(\rho_i - \rho_f) \tag{4}$$

where ρ_f is the fluid density, ρ_i is the density of species i, and ρ_m is the mixture density defined by:

$$\rho_m = \left(\rho_f(1-\phi) + \sum_{j=1}^{i-1} \rho_j \phi_j\right) \bigg/ \left(1 - \sum_{j=i}^{N} \phi_j\right). \tag{5}$$

Note that the particle species are numbered in ascending order by sedimentation velocity.

A rigorous theory for predicting sedimentation velocities in polydisperse dilute suspensions of spheres at low Reynolds number was developed by Batchelor (1982):

$$f_i = 1 + \sum_{j=1}^{N} S_{ij}\phi_j \tag{6}$$

correct to order ϕ. The dimensionless sedimentation coefficients, S_{ij}, are generally negative and are functions of the radius ratio, $\lambda = a_j/a_i$, the reduced density ratio, $\gamma = (\rho_j - \rho_f)/(\rho_i - \rho_f)$, the Péclet number based on the relative Stokes velocity and diffusivity and the interparticle attractive or repulsive forces. Numerical results for large and small Péclet numbers over a wide range of λ and γ values are provided by Batchelor and Wen (1982). The predicted sedimentation velocities are in good agreement with data for dilute monodisperse and bidisperse colloidal suspensions (Cheng and Schachman, 1955; Buscall et al., 1982; Al-Naafa and Selim, 1992) and for dilute mono-

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disperse, bidisperse and tridisperse noncolloidal suspensions (Davis and Birdsell, 1988). Unfortunately, this theory fails for concentrated suspensions.

Proposed Hindered Settling Function

A new hindered settling function with no empirical parameters is proposed for polydisperse suspensions of spheres with arbitrary sizes, densities and concentrations at low Reynolds number:

$$f_i = (1-\phi)^{-S_{ii}}\left(1+\sum_{j\neq i}(S_{ij}-S_{ii})\phi_j\right) \tag{7}$$

where $\phi = \Sigma\phi_j$ is the total particle volume fraction. The key features of this expression are that it agrees with the Batchelor (1982) theory (Eq. 6) in the dilute limit ($\phi \ll 1$), that its only parameters are the sedimentation coefficients determined from Batchelor's theory and that it reduces to the Richardson and Zaki (1954) correlation (Eq. 3) for monodisperse suspensions, with n replaced by $-S_{ii}$. Values cited by Batchelor and Wen (1982) are $S_{ii} = -6.5$ for $\lambda \approx 1$ and $\gamma \approx 1$ at small Péclet number, $S_{ii} = -5.6$ for $\lambda \approx 1$ and $\gamma = 1$ at large Péclet number, and $S_{ii} = -2.6$ for $\lambda = 1$ and $\gamma \approx 1$ at large Péclet number. As noted by Russel et al. (1989), a composite value of $-S_{ii}$ for nearly monodisperse, noncolloidal suspensions with small density and size variations is close to the empirical value of $n = 5.1$.

Comparison with Bidisperse Settling Data

In order to test the proposed hindered settling function, we compare it and selected previous hindered settling functions to data obtained for bidisperse suspensions by a variety of investigators. An initially well-mixed bidisperse suspension will divide into two regions; the lower region contains both particle species at their initial concentrations, and the upper region contains only the slower-settling species. The common procedure is to measure by visual, optical or acoustical means the settling rates of the interface separating the upper and lower regions and the interface separating the upper region from clarified fluid above. The former is denoted by $v_{1,1}$ and the latter by $v_{2,2}$, where the first subscript refers to the particle species and the second subscript refers to the region. It is important to note that mass continuity requires that the volume fraction of species 2 is larger in the upper region than in the lower region (Smith, 1966):

$$\phi_{2,2} = \phi_{2,1}(v_{1,1}-v_{2,1})/(v_{1,1}-v_{2,2}). \tag{8}$$

Table 1 summarizes the properties of fluid and spherical particles used in several previous studies of bidisperse suspensions. The Reynolds and Péclet numbers are defined by $Re_1 = 2\rho_f a_1 u_1^o/\mu_f$ and $Pe_{12} = (a_1+a_2)(u_1^o - u_2^o)/2D_{22}^o$, respectively. Since in all cases $Re_1 \ll 1$, the isolated settling velocity and relative diffusivity are, respectively (Batchelor, 1982):

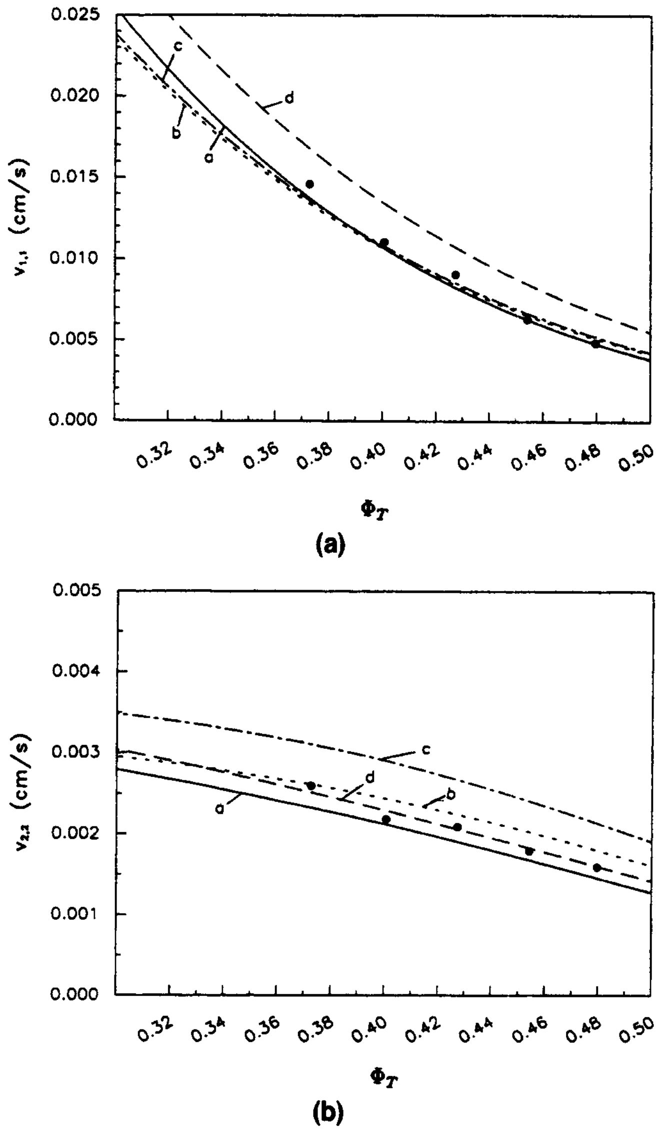

Figure 1. Fall speeds of lower ($v_{1,1}$) and upper ($v_{2,2}$) interfaces vs. initial total particle volume fraction for bidiperse batch sedimentation experiments by Mirza and Richardson (1979).

In these experiments, $\phi_{2,1} = 0.156$ was fixed, whereas $\phi_{1,1}$ was varied. The symbols are the measured data, and the curves are model predictions using the following hindered settling expressions: (a) Eq. 7, (b) Eq. 4 with $n = 5.1$, (c) Eq. 3a with $n = 5.1$, (d) Eq. 3b with $n = 5.1$.

Table 1. Properties of Fluid and Particles for Bidisperse Experiments

Reference	μ_f kg/m·s	ρ_f kg/m³	ρ_1 kg/m³	ρ_2 kg/m³	$2a_1$ 10^{-6} m	$2a_2$ 10^{-6} m	Re_1	Pe_{12}	S_{12}	S_{21}	S_{11}	S_{22}
Mirza and Richardson (1979)	0.1618	909	2,950	2,950	462	115	4×10^{-4}	7×10^{8}	−3.8	−24.5	−5.6	−5.6
Kothari (1981)	0.0184	1,108	2,430	2,430	460	137	0.23	5×10^{9}	−3.9	−18.8	−5.6	−5.6
Al-Naafa and Selim (1989)	0.0274	1,113	2,880	2,500	274	81	7×10^{-4}	9×10^{8}	−3.4	−26.7	−5.6	−5.6
Davis and Birdsell (1988)	0.086	1,015	2,490	2,490	261	136	2×10^{-3}	8×10^{8}	−3.9	−19.0	−5.6	−5.6
Al-Naafa and Selim (1992)	0.0009	775	2,061	1,924	0.484	0.261	8×10^{-8}	9×10^{-3}	−4.5	−12.3	−6.5	−6.5

$$u_i^o = \frac{2}{9}\frac{(\rho_i - \rho_f)a_i^2 g}{\mu_f}, \quad D_{ij}^o = \frac{(a_i + a_j)kT}{6\pi\mu_f a_i a_j} \qquad \text{(9a,b)}$$

where $k = 1.38 \times 20^{-23}$ J/K is the Boltzmann constant, T is the absolute temperature, $g = 9.81$ m/s^2 is the gravitational acceleration and μ_f is the fluid viscosity. Table 1 also includes values for the sedimentation coefficients taken from Batchelor and Wen (1982). For noncolloidal particles ($Pe \gg 1$), we choose $S_{11} = S_{22} = -5.6$, since it is assumed that each species has fixed density with sizes which vary by a few percent (Davis and Birdsell, 1988). For colloidal particles ($Pe \ll 1$), we choose $S_{11} = S_{22} = -6.5$.

Figures 1-3 show the measured and predicted results for the first three studies listed in Table 1. These studies employed concentrated suspensions of noncolloidal particles. The predicted value of the sedimentation velocity $v_{1,1}$ of the lower interface was determined by using Eq. 3a, 6 or 7 together with Eq. 1a, or by using Eq. 3b or 4 together with Eqs. 1b and 2. The sedimentation velocity $v_{2,1}$ of the smaller particles in the lower region was determined in a similar fashion, and then Eq. 8 was solved for $\phi_{2,2}$. The latter was then used with the chosen hindered settling function to predict the sedimentation velocity $v_{2,2}$ of the upper interface. The proposed hindered settling function, Eq. 7, provides a good prediction of the

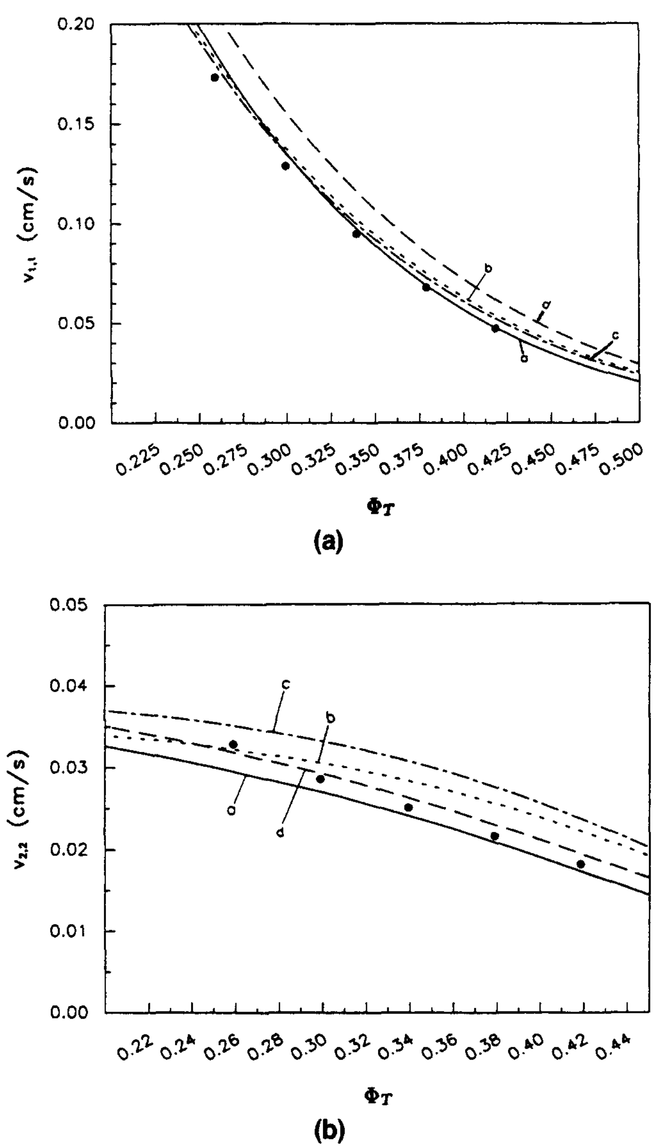

Figure 2. Fall speeds of lower ($v_{1,1}$) and upper ($v_{2,2}$) interfaces vs. initial total particle volume fraction for bidisperse batch sedimentation experiments by Kothari (1981).

In these experiments, $\phi_{2,1} = 0.119$ was fixed, whereas $\phi_{1,1}$ was varied. The symbols are the measured data, and the curves are model predictions using the following hindered settling expressions: (a) Eq. 7, (b) Eq. 4 with $n = 5.1$, (c) Eq. 3a with $n = 5.1$, (d) Eq. 3b with $n = 5.1$.

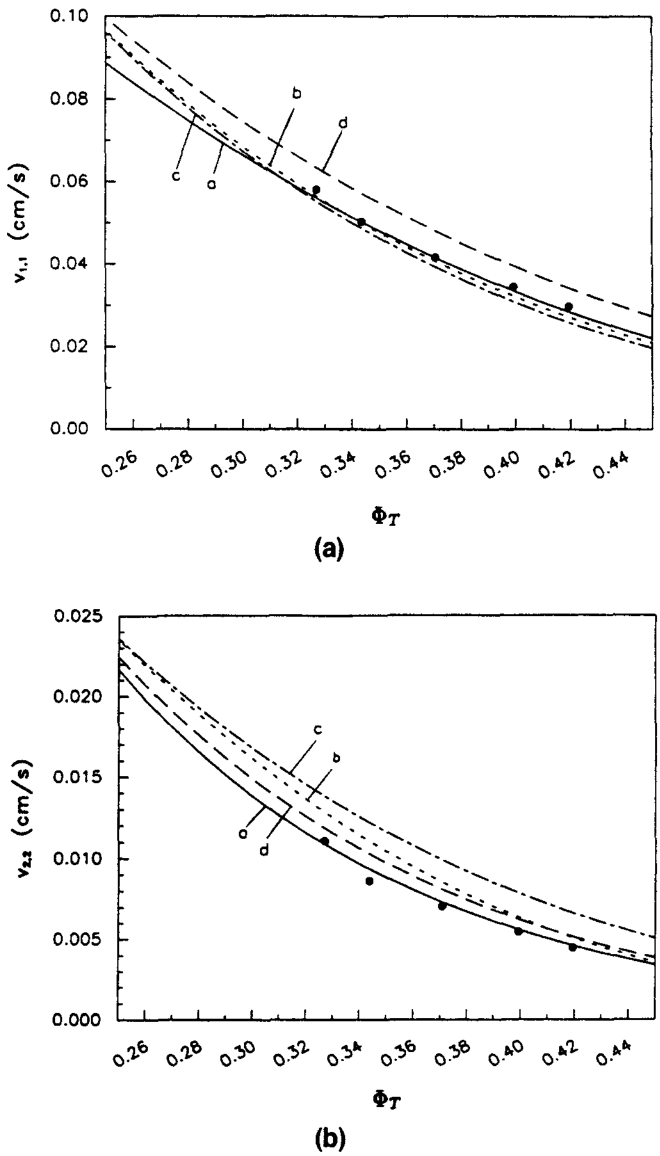

Figure 3. Fall speeds of lower ($v_{1,1}$) and upper ($v_{2,2}$) interfaces vs. initial total particle volume fraction for bidisperse batch sedimentation experiments by Al-Naafa and Selim (1989).

In these experiments, $\phi_{1,1} = 0.220$ was fixed, whereas $\phi_{2,1}$ was varied. The symbols are the measured data, and the curves are model predictions using the following hindered settling expressions: (a) Eq. 7, (b) Eq. 4 with $n = 5.1$, (c) Eq. 3a with $n = 5.1$, (d) Eq. 3b with $n = 5.1$.

settling velocities of both interfaces for all three sets of experiments. The average absolute relative difference between the predicted and measured velocities is only 5% using Eq. 7 for the three sets of data combined. This difference is 10% using Eq. 4, 13% using Eq. 3b, and 15% using Eq. 3a.

Figure 4 shows similar results for the semidilute experiments reported by Davis and Birdsell (1988) for noncolloidal particles. The dilute theory given by Eq. 6 gives good agreement with the data when the total particle volume fraction is less than about 0.08. As the particle concentration increases, Eq. 6 increasingly underpredicts the velocities of the upper and lower interfaces, whereas the proposed hindered settling function (Eq. 7) continues to yield close agreement with the data.

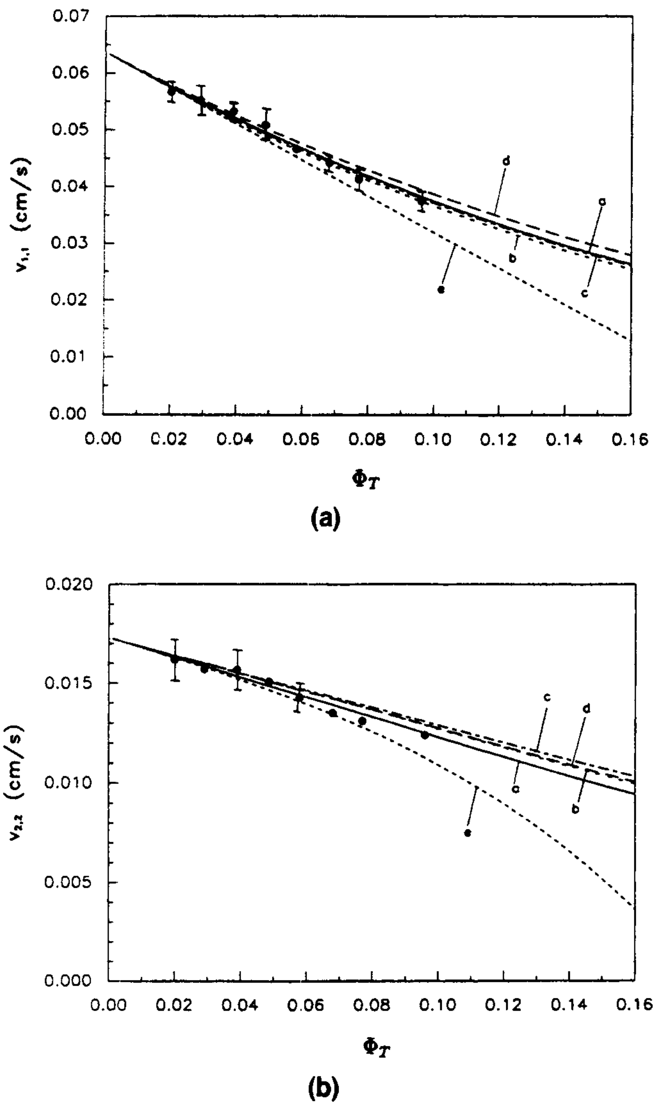

Figure 4. Fall speeds of lower ($v_{1,1}$) and upper ($v_{2,2}$) interfaces vs. initial total particle volume fraction for bidisperse batch sedimentation experiments by Davis and Birdsell (1988).

In these experiments, $\phi_{1,1} = \phi_{2,1}$ were both varied. The symbols are the measured data, and the curves are model predictions using the following hindered settling expressions: (a) Eq. 7, (b) Eq. 4 with $n = 5.1$, (c) Eq. 3a with $n = 5.1$, (d) Eq. 3b with $n = 5.1$, (e) Eq. 6. The error bars are plus and minus one standard deviation for several repeated experiments.

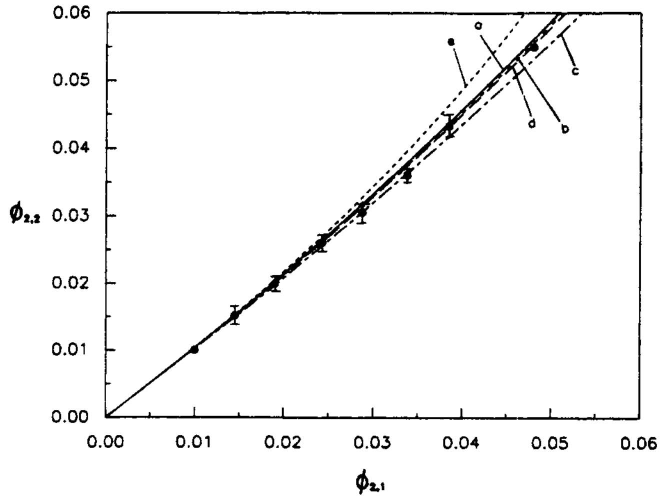

Figure 5. Volume fraction of small particles in the upper region vs. their volume fraction in the lower region, for the experiments by Davis and Birdsell (1988) in Figure 4.

Davis and Birdsell (1988) used a light extinction technique to measure the concentration of the slower-settling particles in the upper region. As shown in Figure 5, using Eq. 8 together with any of the hindered settling functions listed (except Eq. 6 for the higher concentrations) predicts accurately the measured relative increase in the concentration of slower settling particles in the upper region with increasing concentration in the lower region.

Finally, Figure 6 shows the measured and predicted velocities of the upper and lower interfaces for bidisperse sedimentation experiments performed by Al-Naafa and Selim (1992) with colloidal particles. The proposed hindered settling function (Eq. 7) slightly overpredicts the settling velocity of the lower interface, but it generally provides much better agreement than any of the previous correlations. Also showing very good agreement is a model proposed by Al-Naafa and Selim (1992) which is identical to Eq. 3b, except that the value of $n = 5.1$ for noncolloidal particles is replaced by $n = 6.5$ for colloidal particles.

Concluding Remarks

A new hindered settling function containing no empirical parameters has been proposed for the sedimentation of polydisperse suspensions at low Reynolds number. This function has the form of the Richardson-Zaki formula for monodisperse suspensions and agrees with the theory of Batchelor (1982) for dilute suspensions. Its agreement with bidisperse sedimentation experiments is equal to or better than that of the common formulae tested which have been adapted from monodisperse correlations. It is recommended that the new function be tested over a broader range of particle sizes and densities, and for tridisperse and more complex suspensions.

The only parameters appearing in the proposed expression are the sedimentation coefficients calculated by Batchelor and Wen (1982) based on two-sphere hydrodynamic theory. For convenience, we quote some of their key findings: (i) $S_{ij} \rightarrow -2.5 - \gamma$ for $\lambda \ll 1$ and arbitrary Pe_{ij}; (ii) $S_{ij} \rightarrow -\gamma(\lambda^2 + 3\lambda + 1)$ for $\lambda \gg 1$ and arbitrary Pe_{ij}; (iii) $S_{ij} = -2.52 - 0.13\gamma$ for $\lambda = 1$,

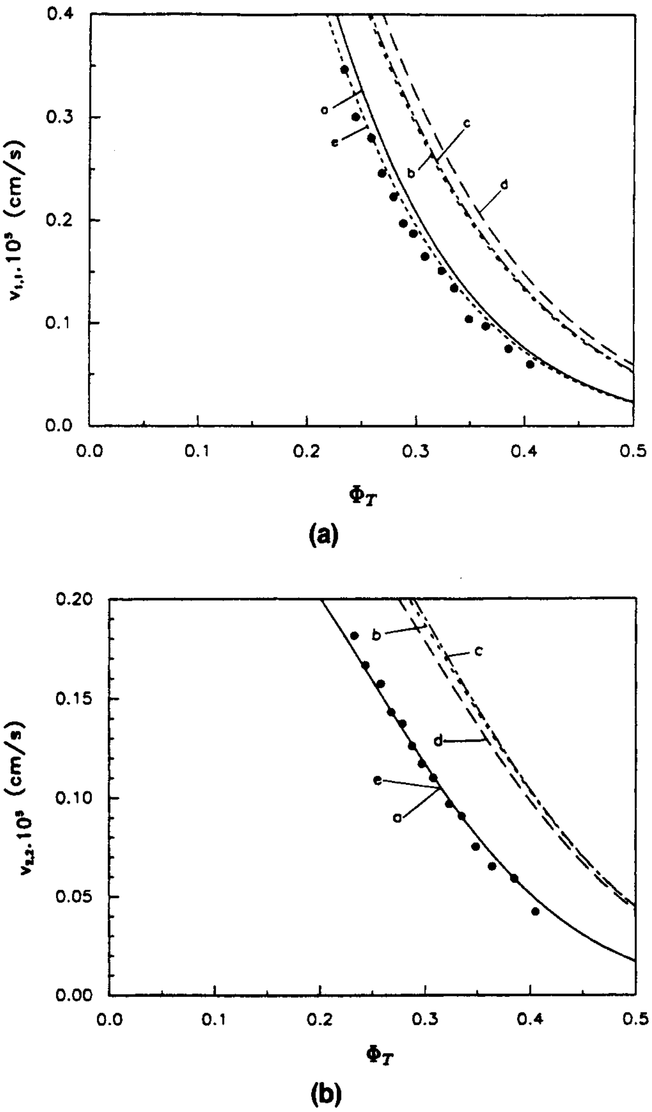

Figure 6. Fall speeds of lower ($v_{1,1}$) and upper ($v_{2,2}$) interfaces vs. initial total particle volume fraction for bidisperse batch sedimentation experiments by Al-Naafa and Selim (1992).

In these experiments, $\phi_{2,1}=0.079$ was fixed, whereas $\phi_{1,1}$ was varied. The symbols are the measured data, and the curves are model predictions using the following hindered settling expressions: (a) Eq. 7, (b) Eq. 4 with $n=5.1$, (c) Eq. 3a with $n=5.1$, (d) Eq. 3b with $n=5.1$, (e) Eq. 3b with $n=6.5$.

$\gamma \neq 1$, and $Pe_{ij} \gg 1$. In these expressions, $\lambda = a_j/a_i$ and $\gamma = (\rho_j-\rho_f)/(\rho_i-\rho_f)$. For the common case of equidensity particles ($\gamma=1$), we fit the numerical data of Batchelor and Wen (1982) over the range $0 \leq \lambda \leq 8$ to polynomials:

$$S_{ij} = -3.50 - 1.10\lambda - 1.02\lambda^2 - 0.002\lambda^3 \quad Pe_{ij} \gg 1, \tag{10}$$

$$S_{ij} = -3.42 - 1.96\lambda - 1.21\lambda^2 - 0.013\lambda^3 \quad Pe_{ij} \ll 1. \tag{11}$$

Finally, we note that the proposed hindered settling function does not apply for suspensions in which lateral segregation occurs so that vertical fingers and blobs form which are convected at a rapid rate due to their buoyancy (Whitmore, 1955). Lateral segregation has been observed in relatively concentrated bidisperse suspensions with large differences in the densities or sizes of the two types of particles (Weiland et al., 1984). A stability analysis providing a theoretical basis for this phenomenon has been proposed by Batchelor and Janse van Rensburg (1986).

Notation

- a = particle radius
- D^o = Brownian diffusion coefficient for isolated spheres
- f = hindered settling function based on particle settling velocity
- f^s = hindered settling function based on particle slip velocity
- g = gravitational acceleration
- k = Boltzmann constant
- n = Richardson-Zaki exponent
- N = number of particle species
- Pe = Péclet number
- Re = Reynolds number
- S_{ij} = sedimentation coefficient
- T = absolute temperature
- u^o = Stokes settling velocity of an isolated sphere
- v = hindered particle settling velocity relative to bulk suspension
- v^s = hindered particle slip velocity relative to fluid

Greek letters

- γ = relative density ratio, $(\rho_j-\rho_f)/(\rho_i-\rho_f)$
- λ = radius ratio, a_j/a_i
- μ_f = fluid viscosity
- ρ_f = fluid density
- ρ_i = density of particle species i
- ρ_m = mixture density
- ϕ = total particle volume fraction
- ϕ_i = volume fraction of particle species i

Subscripts

- i, j = particle species
- 1 = faster-settling particles (first subscript); lower zone (second subscript)
- 2 = slower-settling particles (first subscript); upper zone (second subscript)

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